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ON CONSTRUCTION OF APPROXIMATE SOLUTIONS OF EQUATIONS OF THE SHALLOW SPHERICAL SHELL*

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Abstract—The new theory of the elastic shells leads to the elliptic system of equations of the 12th order (see I. N. Vekua, *The Theory of Thin Shallow Shells with Variable Thickness*, Tbilisi, Metsniereba, 1965). In the case of the spherical shell, the solutions of this system of equations may be exactly expressed by six functions w_i satisfying the equations of the form $\nabla^2 w + k_i^2 w = 0$ ($i = 1, 2, 3, 4, 5$) where $k_1^2, k_2^2, k_4^2, k_5^2$ are real constants, k_3^2 is a complex constant, w_1, w_2, w_4, w_5 are real-valued functions, and w_3 is a complex-valued function. These functions are expressed by six arbitrary analytic functions of one complex variable. For the shallow spherical shells the obtained formulae may be essentially simplified. The same method may be also used for the simplification of the equations of the shallow cylindrical shells.

I

WE SHALL consider the version of the theory of elastic shells according to which the displacement vector $\mathbf{U}(x^1, x^2, x^3)$ of the point (x^1, x^2, x^3) $(-h \leq x^3 \leq h)$, $2h$ is the thickness of the shell, and the stress force $\mathbf{P}_{(l)}(x^1, x^2, x^3)$ acting on the cross section Σ_l with the unit normal $\mathbf{l}$ are expressed by the following formulae (Approximations of the order $N = 1$; see [1, 2])

$$\mathbf{U}(x^1, x^2, x^3) = \mathbf{u}(x^1, x^2) + \frac{x^3}{2h}\mathbf{v}(x^1, x^2), \tag{1}$$

$$\mathbf{P}_{(l)}(x^1, x^2, x^3) = \frac{1}{2h}\mathbf{T}_{(l)}(x^1, x^2) + \frac{3x^3}{2(h)^3}\mathbf{S}_{(l)}(x^1, x^2), \tag{2}$$

or

$$\mathbf{P}_{(l)} = \frac{1}{2h}\mathbf{T}_{(l)} + \frac{3x^3}{2(h)^3}\mathbf{M}_{(l)} \times \mathbf{n} + \frac{2x^3}{(h)^2} Q_{(l)}\mathbf{n}. \tag{3}$$

Here $\mathbf{u}(x^1, x^2)$ and $\mathbf{v}(x^1, x^2)$ are the vector fields on the middle surface $x^3 = 0$, $\mathbf{T}_{(l)}$ is the stress resultant, $\mathbf{M}_{(l)}$ is the stress couple, $Q_{(l)}$ is the force of the self-equilibrated transverse couple $(Q_{(l)}\mathbf{n}, -Q_{(l)}\mathbf{n})$ (see Fig. 1); $\mathbf{n}$ is the unit normal to the middle surface $x^3 = 0$

In the case of the shallow or thin shell $\mathbf{T}_{(l)}$ and $\mathbf{S}_{(l)}$ are expressed by $\mathbf{u}$ and $\mathbf{v}$ by means of the following formulae:

$$\mathbf{T}_{(l)} = 2h\left[\lambda\left(\left(\mathbf{z}^\alpha \frac{\partial \mathbf{u}}{\partial x^\alpha}\right) + (\mathbf{n}\mathbf{v})\right)\mathbf{l} + \mu\frac{\partial \mathbf{u}}{\partial l} + \mu\left(\mathbf{l}\frac{\partial \mathbf{u}}{\partial x^\alpha}\right)\mathbf{z}^\alpha + \mu(\mathbf{l}\mathbf{v})\mathbf{n}\right], \tag{4}$$

$$\mathbf{S}_{(l)} = \frac{4(h)^2}{3}\left[\lambda\left(\mathbf{z}^\alpha \frac{\partial \mathbf{v}}{\partial x^\alpha}\right)\mathbf{l} + \mu\frac{\partial \mathbf{v}}{\partial l} + \mu\left(\mathbf{l}\frac{\partial \mathbf{v}}{\partial x^\alpha}\right)\mathbf{z}^\alpha\right], \tag{5}$$

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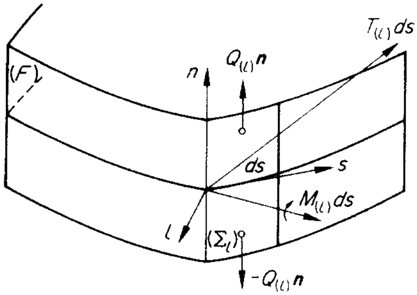

FIG. 1.

where λ and μ are Lamé's constants,

$$\lambda = \frac{E\sigma}{(1+\sigma)(1-2\sigma)}, \qquad \mu = \frac{E}{2(1+\sigma)}. \tag{6}$$

Here E is Young's modulus and σ is Poisson's ratio; $\mathbf{z}^1, \mathbf{z}^2$ are the conjugate basic vectors of the coordinate system x^1, x^2 on the middle surface $x^3 = 0$.

II

The above suggested scheme of distribution of stresses and strains in the shell leads to the elliptic system of differential equations of the twelfth order (see [1, 2]). For plates and spherical shells with constant thicknesses all solutions of these equations may be represented by formulae which contain six arbitrary analytic functions of one complex variable. This fact allows us to apply the theory of analytic functions to static problems of elastic plates and spherical shells (see [1–3]).

In the case of the plate these formulae have comparatively simple structure. The similar forms have also the formulae for spherical shells (see [1]). But the dependence on arbitrary analytic functions is realised there by means of some integral transformations and due to this the application of these formulae is considerably complicated. Besides the main terms, defining the essential parts of the stress–strain distribution picture, the formulae contain many insignificant ones which cause different mathematical complications. Therefore it is practically very important to simplify beforehand the system of equations neglecting the small quantities which have no essential mechanical influence on the picture of stress–strain distribution. For this purpose one can suggest different approaches (see [1, 5]). It is worth noting that, in general, the different ways lead to different results. Below we shall apply an approach for simplification of the system of equations of the shallow spherical shell. We shall also use the method of expansion of solutions in the power series with respect to the small parameter

$$\varepsilon = \frac{2h}{R}, \tag{7}$$

where R is the radius of the sphere. This method allows the reduction of static problems for elastic spherical shells to the sequences of similar problems for the elastic plate, the thickness of which is equal to 1. It is worth noting that similar approaches may be also used for any shell, e.g. for cylindrical ones.

III

Let us consider the isometric coordinates on the sphere

$$\xi = \operatorname{tg}\frac{\theta}{2}\cos\varphi, \qquad \eta = \operatorname{tg}\frac{\theta}{2}\sin\varphi,$$

where θ and φ are the geographical coordinates. For the shallow spherical shell the coordinate θ varies inside the small segment: $0 \leq \theta \leq \theta_0$. Therefore one can put

$$\xi = \tfrac{1}{2}\theta\cos\varphi, \qquad \eta = \tfrac{1}{2}\theta\sin\varphi.$$

Further it will be more convenient to consider the following new coordinates

$$x = \frac{R}{2h}\theta\cos\varphi, \qquad y = \frac{R}{2h}\theta\sin\varphi. \tag{8}$$

Then for the metric quadratic form we obtain the formula

$$\mathrm{d}s^2 = (2h)^2(\mathrm{d}x^2+\mathrm{d}y^2) = (2h)^2\,\mathrm{d}z\,\mathrm{d}\bar{z}, \tag{9}$$

where

$$z = x+iy, \qquad \bar{z} = x-iy.$$

Now the system of equations of the elastic shell may be written in the following complex form (see [1, 2])

$$\begin{aligned}
&\frac{\partial}{\partial z}(T_1-T_2+2iS)+\frac{\partial}{\partial \bar{z}}(T_1+T_2)-\varepsilon(N_1+iN_2)+X_1+iX_2 = 0,\\
&\frac{\partial}{\partial z}(N_1+iN_2)+\frac{\partial}{\partial \bar{z}}(N_1-iN_2)+\varepsilon(T_1+T_2)+X = 0,\\
&\frac{\partial}{\partial z}(M_1-M_2-2iH)+\frac{\partial}{\partial \bar{z}}(M_1+M_2)-\varepsilon(S_1+iS_2)-2h(N_1+iN_2) = 0,\\
&\frac{\partial}{\partial z}(S_1+iS_2)+\frac{\partial}{\partial \bar{z}}(S_1-iS_2)+\varepsilon(M_1+M_2)-2hT = 0,
\end{aligned} \tag{10}$$

where X_1, X_2 and X are components of external loads,

$$T_1-T_2+2iS = 4\mu\frac{\partial W_1}{\partial \bar{z}}, \qquad W_1 = u_1+iu_2,$$

$$T_1+T_2 = 2(\lambda+\mu)\theta_1+2\lambda v-4\varepsilon(\lambda+\mu)u, \qquad \theta_1 = \frac{\partial u_1}{\partial x}+\frac{\partial u_2}{\partial y},$$

$$N_1+iN_2 = \mu\left(2\frac{\partial u}{\partial \bar{z}}+W_2+\varepsilon W_1\right), \qquad W_2 = v_1+iv_2, \tag{11}$$

$$
\begin{aligned}
&T = \lambda\theta_1 + (\lambda+2\mu)v - 2\varepsilon\lambda u, \\
&M_1 - M_2 - 2iH = \frac{2\mu h}{3}\frac{\partial W_2}{\partial \bar{z}}, \\
&M_1 + M_2 = \frac{(\lambda+\mu)h}{3}(\theta_2 - 2\varepsilon v), \qquad \theta_2 = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y}, \\
&S_1 + iS_2 = \frac{\mu h}{6}\left(2\frac{\partial v}{\partial \bar{z}} + \varepsilon W_2\right), \\
&\left[\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\right].
\end{aligned} \tag{11}
$$

Here we use the following notation: T_1 and T_2 are the normal forces, S is the shearing force, M_1 and M_2 are the bending moments, H is the twisting moment, N_1 and N_2 are the transverse forces, S_1 and S_2 are connected with the so called splitting forces Q_1 and Q_2 by the formulae (see [1, 2])

$$
S_1 = \frac{4}{3h}Q_1, \qquad S_2 = \frac{4}{3h}Q_2. \tag{12}
$$

Further

$$
W_1 = u_1 + iu_2, \qquad W_2 = v_1 + iv_2 \tag{13}
$$

$$
\theta_1 = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} = \frac{\partial W_1}{\partial z} + \frac{\partial \overline{W}_1}{\partial \bar{z}}, \tag{14}
$$

$$
\theta_2 = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} = \frac{\partial W_2}{\partial z} + \frac{\partial \overline{W}_2}{\partial \bar{z}}, \tag{15}
$$

where u_1, u_2 and v_1, v_2 are the tangent components of the vectors $\mathbf{u}$ and $\mathbf{v}$, respectively; u and v denote the normal components of these vectors. We propose that the unit normal $\mathbf{n}$ to the sphere is directed to the centre.

By substituting (11) into (10) we obtain the elliptic system of equations of the twelfth order which may be written with respect to the real variables x and y in the following form

$$
\begin{aligned}
&\mu\Delta u_1 + (\lambda+\mu)\frac{\partial\theta_1}{\partial x} + \lambda\frac{\partial v}{\partial x} - \varepsilon\left(\mu v_1 + (2\lambda+3\mu)\frac{\partial u}{\partial x}\right) - \varepsilon^2\mu u_1 + X_1 = 0, \\
&\mu\Delta u_2 + (\lambda+\mu)\frac{\partial\theta_1}{\partial y} + \lambda\frac{\partial v}{\partial y} - \varepsilon\left(\mu v_2 + (2\lambda+3\mu)\frac{\partial u}{\partial y}\right) - \varepsilon^2\mu u_2 + X_2 = 0, \\
&\mu\Delta u + \mu\theta_2 + \varepsilon[2\lambda v + (2\lambda+3\mu)\theta_1] - 4\varepsilon^2(\lambda+\mu)u + X = 0, \\
&\mu\Delta v_1 + (\lambda+\mu)\frac{\partial\theta_2}{\partial x} - 12\mu\left(\frac{\partial u}{\partial x} + v_1\right) - \varepsilon\left(12\mu u_1 + (2\lambda+3\mu)\frac{\partial v}{\partial x}\right) - \varepsilon^2\mu v_1 = 0, \\
&\mu\Delta v_2 + (\lambda+\mu)\frac{\partial\theta_2}{\partial y} - 12\mu\left(\frac{\partial u}{\partial y} + v_2\right) - \varepsilon\left(12\mu u_2 + (2\lambda+3\mu)\frac{\partial v}{\partial y}\right) - \varepsilon^2\mu v_2 = 0, \\
&\mu\Delta v - 12[\lambda\theta_1 + (\lambda+2\mu)v] + \varepsilon[24\lambda u + (2\lambda+3\mu)\theta_2] - 4\varepsilon^2(\lambda+\mu)v = 0.
\end{aligned} \tag{16}
$$

It is easy to derive the formulae expressing all solutions of this system by six functions satisfying the equations of the form

$$\Delta W - K^2 W = 0, \qquad (K^2 = \text{const})$$

But it is more convenient to use a procedure based on the power series expansions with respect to the small parameter ε.

IV

If $\varepsilon = 0$ then we obtain two independent systems of equations which may be written in the following form

$$\begin{aligned} &\mu\Delta u_1 + (\lambda+\mu)\frac{\partial\theta_1}{\partial x} + \lambda\frac{\partial v}{\partial x} + X_1 = 0,\\ &\mu\Delta u_2 + (\lambda+\mu)\frac{\partial\theta_1}{\partial y} + \lambda\frac{\partial v}{\partial y} + X_2 = 0,\\ &\mu\Delta v - 12[\lambda\theta_1 + (\lambda+2\mu)v] = 0. \end{aligned} \tag{17}$$

$$\begin{aligned} &\mu\Delta v_1 + (\lambda+\mu)\frac{\partial\theta_2}{\partial x} - 12\mu\left(\frac{\partial u}{\partial x} + v_1\right) = 0,\\ &\mu\Delta v_2 + (\lambda+\mu)\frac{\partial\theta_2}{\partial y} - 12\mu\left(\frac{\partial u}{\partial y} + v_2\right) = 0,\\ &\mu\Delta u + \mu\theta_2 + X = 0. \end{aligned} \tag{18}$$

These systems of equations coincide with those for the plate the thickness of which is equal to 1 (see [1, 2]). Therefore one can write down the formulae expressing in the exact form all solutions of equations (17) and (18) by means of arbitrary analytic functions.

For equations (17) these formulae may be written in the following form (see [1, 2])

$$\begin{aligned} u_1 &= \frac{\partial U}{\partial x} - \frac{\partial U_*}{\partial y} + u_1^0,\\ u_2 &= \frac{\partial U}{\partial y} + \frac{\partial U_*}{\partial x} + u_2^0, \end{aligned} \tag{19}$$

$$v = -\frac{\sigma}{1-\sigma}\Delta U - \frac{1-2\sigma}{24\sigma}\Delta\Delta U + v^0, \tag{20}$$

where u_1^0, u_2^0, v^0 is a particular solution of the non-homogeneous equations (17),

$$U = -\frac{\sigma}{24}\chi + \frac{1-\sigma}{2(1+\sigma)}[zf'-(z) + \bar{z}f'(z) - \tfrac{1}{2}(f'_0(z) + \overline{f'_0(z)})], \tag{21}$$

$$U_* = \frac{i}{1+\sigma}[z\overline{f'(z)} - \bar{z}f'(z)]. \tag{22}$$

Here f_0 and f are the arbitrary analytic functions of z and χ is the arbitrary real solution of the equation

$$\Delta\chi-\frac{24}{1-\sigma}\chi = 0. \tag{23}$$

All solutions of equation (23) may be represented by the formula (see [4])

$$\chi = \mathrm{Re}\left[f_1(z)-\int_0^z f_1(t)\frac{\partial}{\partial t}\mathbf{I}_0\{\kappa_1\sqrt{[\bar{z}(z-t)]}\}\,\mathrm{d}t\right], \left(\kappa_1^2 = \frac{24}{1-\sigma}\right), \tag{24}$$

where f_1 is the arbitrary analytic function of z, $\mathbf{I}_0$ is the Bessel function of zero order with the imaginary argument.

Therefore the formulae expressing all solutions of equations (17) contain three arbitrary analytic functions f_0, f, f_1.

Sometimes it is more convenient to represent the function U in the following form

$$U = \Delta\tilde{U}, \tag{25}$$

where

$$\tilde{U} = -\frac{\sigma(1-\sigma)}{576}\chi+\frac{1-\sigma}{16(1+\sigma)}[z^2\bar{f}+\bar{z}^2 f-z\bar{f}_0-\bar{z}f_0]. \tag{26}$$

Formulae (20) may also be written in the form

$$v = \chi-\frac{2\sigma}{1+\sigma}(f''+\bar{f}'').$$

All solutions of equation (18) may be represented by the formulae

$$\begin{aligned} v_1 &= \frac{\partial V}{\partial x}-\frac{\partial\psi}{\partial y}, \\ v_2 &= \frac{\partial V}{\partial y}+\frac{\partial\psi}{\partial x}, \end{aligned} \tag{27}$$

$$u = -V+\frac{2(1+\sigma)B}{E}\Delta V, \tag{28}$$

where V and ψ are the arbitrary solutions of the equations

$$\Delta\Delta V = -\frac{1}{B}X, \tag{29}$$

$$\Delta\psi-12\psi = 0. \tag{30}$$

Here

$$B = \frac{1}{12}(\lambda+2\mu)E = \frac{(1-\sigma)E}{12(1+\sigma)(1-2\sigma)}. \tag{31}$$

The function V may be expressed by the formula

$$V = 8[z\overline{g'(z)}+\bar{z}g'(z)]+4[g_0'(z)+\overline{g_0'(z)}]+V^0, \tag{32}$$

where g_0 and g_1 are the arbitrary analytic functions and V^0 is a particular solution of equation (29).

Sometimes it is convenient to represent the function V in the form

$$V = \Delta\tilde{V} + V^0, \tag{33}$$

where

$$\tilde{V} = z^2\overline{g(z)} + \bar{z}^2 g(z) + z\overline{g_0(z)} + \bar{z}g_0(z). \tag{34}$$

All solutions of equation (30) may be represented by the formula

$$\psi = \mathrm{Re}\left[g_1(z) - \int_0^z g_1(t)\frac{\partial}{\partial t}\,\mathbf{I}_0\{\sqrt{[12\bar{z}(z-t)]}\}\,\mathrm{d}t\right].$$

Therefore the formulae representing solutions of equations (18) contain also three arbitrary analytic functions.

V

Let us return now to equations (16) and try to construct the solutions of the form

$$\begin{aligned} u &= \sum_{K=0}^{\infty} u^{(K)}\varepsilon^K, \qquad u_\alpha = \sum_{K=0}^{\infty} u_\alpha^{(K)}\varepsilon^K \qquad (\alpha = 1, 2), \\ v &= \sum_{K=0}^{\infty} v^{(K)}\varepsilon^K, \qquad v_\alpha = \sum_{K=0}^{\infty} v_\alpha^{(K)}\varepsilon^K \qquad (\alpha = 1, 2). \end{aligned} \tag{35}$$

We suppose that $u_\alpha^{(0)}, u^{(0)}, v_\alpha^{(0)}, v^{(0)}$ is an arbitrarily given solution of equations (17) and (18) which may be constructed by means of the formulae given in the previous section. The formal substitution of (35) into (16) shows that the series (35) may satisfy equations (16) if the following equations are fulfilled

$$\begin{aligned} &\mu\Delta u_1^{(K)} + (\lambda+\mu)\frac{\partial\theta_1^{(K)}}{\partial x} + \lambda\frac{\partial v^{(K)}}{\partial x} + X_1^{(K)} = 0, \\ &\mu\Delta u_2^{(K)} + (\lambda+\mu)\frac{\partial\theta_1^{(K)}}{\partial y} + \lambda\frac{\partial v^{(K)}}{\partial y} + X_2^{(K)} = 0, \\ &\mu\Delta v^{(K)} - 12[\lambda\theta_1^{(K)} + (\lambda+2\mu)v^{(K)}] + X_3^{(K)} = 0, \end{aligned} \tag{36}$$

$$\begin{aligned} &\mu\Delta v_1^{(K)} + (\lambda+\mu)\frac{\partial\theta_2^{(K)}}{\partial x} - 12\mu\left(\frac{\partial u^{(K)}}{\partial x} + v_1^{(K)}\right) + X_4^{(K)} = 0, \\ &\mu\Delta v_2^{(K)} + (\lambda+\mu)\frac{\partial\theta_2^{(K)}}{\partial y} - 12\mu\left(\frac{\partial u^{(K)}}{\partial y} + v_2^{(K)}\right) + X_5^{(K)} = 0, \\ &\mu\Delta u^{(K)} + \mu\theta_2^{(K)} + X_6^{(K)} = 0, \\ &(K = 1, 2, \ldots), \end{aligned} \tag{37}$$

where

$$
\begin{aligned}
X_1^{(K)} &= -\mu v_1^{(K-1)} - (2\lambda+3\mu)\frac{\partial u^{(K-1)}}{\partial x} - \mu u_1^{(K-2)},\\
X_2^{(K)} &= -\mu v_2^{(K-1)} - (2\lambda+3\mu)\frac{\partial u^{(K-1)}}{\partial y} - \mu u_2^{(K-2)},\\
X_3^{(K)} &= 24\lambda u^{(K-1)} + (2\lambda+3\mu)\theta_2^{(K-1)} - 4(\lambda+\mu)v^{(K-2)},\\
X_4^{(K)} &= -12\mu u_1^{(K-1)} - (2\lambda+3\mu)\frac{\partial v^{(K-1)}}{\partial x} - \mu v_1^{(K-2)},\\
X_5^{(K)} &= -12\mu u_2^{(K-1)} - (2\lambda+3\mu)\frac{\partial v^{(K-1)}}{\partial y} - \mu v_2^{(K-2)},\\
X_6^{(K)} &= 2\lambda v^{(K-1)} + (2\lambda+3\mu)\theta_1^{(K-1)} - 4(\lambda+\mu)u^{(K-2)},
\end{aligned}
\tag{38}
$$

$$(K = 1, 2, \ldots; \quad u^{(K)} = v^{(K)} = u_\alpha^{(K)} = v_\alpha^{(K)} = 0, \qquad \text{if } K < 0; \quad \alpha = 1, 2).$$

For each fixed K equations (36) and (37) coincide with equations (17) and (18), respectively. Therefore to equations (36) and (37) one can apply the methods of integration stated in the previous section.

Suppose that we consider some linear boundary value problem A_ε for equations (16). The corresponding boundary value problem for equations (17) and (18) will be denoted by A_0. We shall also suppose that the right side parts of the boundary conditions do not depend on the parameter ε. It means that for problems A_ε and A_0 these ones are the same.

Let $u^{(0)}$, $v^{(0)}$, $u_1^{(0)}$, $u_2^{(0)}$, $v_1^{(0)}$, $v_2^{(0)}$ be the solution of the problem A_0. If the series (35) represent the solution of the boundary value problem A_ε, then the solutions of equations (36) and (37) are to satisfy the corresponding homogeneous boundary conditions. Therefore the boundary value problem A_ε for the elastic spherical shell is reduced to the sequence of the boundary value problems for the non-homogeneous systems of equations (36) and (37) of elastic plates with homogeneous boundary conditions. Using the formulae of section IV the latter boundary value problems may be reduced to the problems for analytic functions of one complex variable. It is worth noting that all these problems are of the same type and may be solved using the method requiring repetitions of calculations of the same kind.

VI

For the shallow shell the parameter ε is very small and one can neglect the terms of orders $\varepsilon^m|m \geq 2|$. Therefore the sought solution has the form

$$
\begin{aligned}
u &= u^{(0)} + \varepsilon u^{(1)}, & u_1 &= u_1^{(0)} + \varepsilon u_1^{(1)}, & u_2 &= u_2^{(0)} + \varepsilon u_2^{(1)},\\
v &= v^{(0)} + \varepsilon v^{(1)}, & v_1 &= v_1^{(0)} + \varepsilon v_1^{(1)}, & v_2 &= v_2^{(0)} + \varepsilon v_2^{(1)}.
\end{aligned}
\tag{39}
$$

We assume that $u^{(0)}$, $v^{(0)}$, $u_1^{(0)}$, $u_2^{(0)}$, $v_1^{(0)}$, $v_2^{(0)}$ is the known solution of the boundary value problem A_0.

According to (36)–(38) we will have the equations

$$\mu\Delta u_1^{(1)}+(\lambda+\mu)\frac{\partial\theta_1^{(1)}}{\partial x}+\lambda\frac{\partial v^{(1)}}{\partial x}+X_1^{(1)}=0$$

$$\mu\Delta u_2^{(1)}+(\lambda+\mu)\frac{\partial\theta_1^{(1)}}{\partial y}+\lambda\frac{\partial v^{(1)}}{\partial y}+X_2^{(1)}=0 \tag{40}$$

$$\mu\Delta v^{(1)}-12(\lambda\theta^{(1)}+(\lambda+2\mu)v^{(1)})+X_3^{(1)}=0$$

$$\mu\Delta v_1^{(1)}+(\lambda+\mu)\frac{\partial\theta_2^{(1)}}{\partial x}-12\mu\left(\frac{\partial u^{(1)}}{\partial x}+v_1^{(1)}\right)+X_4^{(1)}=0,$$

$$\mu\Delta v_2^{(1)}+(\lambda+\mu)\frac{\partial\theta_2^{(1)}}{\partial y}-12\mu\left(\frac{\partial u^{(1)}}{\partial y}+v_2^{(1)}\right)+X_5^{(1)}=0, \tag{41}$$

$$\mu\Delta u^{(1)}+\mu\theta_2^{(1)}+X_6^{(1)}=0,$$

where

$$X_1^{(1)}=-(2\lambda+3\mu)\frac{\partial u^{(0)}}{\partial x}-\mu v_1^{(0)},$$

$$X_2^{(1)}=-(2\lambda+3\mu)\frac{\partial u^{(0)}}{\partial y}-\mu v_2^{(0)}, \tag{42}$$

$$X_3^{(1)}=24\lambda v^{(0)}+(2\lambda+3\mu)\theta_2^{(0)},$$

$$X_4^{(1)}=-(2\lambda+3\mu)\frac{\partial v^{(0)}}{\partial x}-12\mu u_1^{(0)},$$

$$X_5^{(1)}=-(2\lambda+3\mu)\frac{\partial v^{(0)}}{\partial y}-12\mu u_2^{(0)},$$

$$X_6^{(1)}=2\lambda v^{(0)}+(2\lambda+3\mu)\theta_1^{(0)}, \tag{43}$$

$$\left(\theta_1^{(1)}=\frac{\partial u_1^{(1)}}{\partial x}+\frac{\partial u_2^{(1)}}{\partial y},\qquad \theta_2^{(1)}=\frac{\partial v_1^{(1)}}{\partial x}+\frac{\partial v_2^{(1)}}{\partial y}\right).$$

We shall suppose that the external loads are absent, i.e. $X_1=X_2=X=0$. The general case may be always reduced to this special one. Therefore one can put $u_1^0=u_2^0=u^0=0$, $v^0=0$ and according to the formulae of section IV, the solution of the problem A_0 may be represented in the following form

$$u_1^{(0)}=\frac{\partial\Delta\tilde{U}}{\partial x}-\frac{\partial U_*}{\partial y},\qquad u_2^{(0)}=\frac{\partial\Delta\tilde{U}}{\partial y}+\frac{\partial U_*}{\partial x},$$

$$v^{(0)}=-\frac{\sigma}{1-\sigma}\Delta\Delta\tilde{U}-\frac{1-2\sigma}{24\sigma}\Delta\Delta\Delta\tilde{U}, \tag{44}$$

$$v_1^{(0)}=\frac{\partial\Delta\tilde{V}}{\partial x}-\frac{\partial\psi}{\partial y},\qquad v_2^{(0)}=\frac{\partial\Delta\tilde{V}}{\partial y}+\frac{\partial\psi}{\partial x},$$

$$u^{(0)}=-\Delta\tilde{V}+\frac{2(1+\sigma)B}{E}\Delta\Delta\tilde{V}, \tag{45}$$

where

$$\tilde{U} = -\frac{\sigma(1-\sigma)}{576}\chi + \frac{1-\sigma}{16(1+\sigma)}(z^2\bar{f} + \bar{z}^2 f - z\bar{f}_0 - zf_0), \tag{46}$$

$$U_* = \frac{i}{1+\sigma}(z\bar{f}' - \bar{z}f'), \tag{47}$$

$$\tilde{V} = z^2\bar{g} + \bar{z}^2 g + z\bar{g}_0 + \bar{z}g_0. \tag{48}$$

Here f_0, f, g_0, g are analytic functions, χ and ψ are solutions of equations (23) and (30).

According to (42) and (45) one can write

$$\begin{aligned} X_1^{(1)} &= \frac{\partial}{\partial x}(a_0\Delta\tilde{V} + a_1\Delta\Delta\tilde{V}) + \frac{\partial\mu\psi}{\partial y}, \\ X_2^{(1)} &= \frac{\partial}{\partial y}(a_0\Delta\tilde{V} + a_1\Delta\Delta\tilde{V}) - \frac{\partial\mu\psi}{\partial x}, \\ X_3^{(1)} &= b_0\Delta\tilde{V} + b_1\Delta\Delta\tilde{V}. \end{aligned} \tag{49}$$

$$\begin{aligned} X_4^{(1)} &= \frac{\partial}{\partial x}(c_0\Delta\tilde{U} + c_1\Delta\Delta\tilde{U} + c_2\Delta\Delta\Delta\tilde{U}) + \frac{\partial}{\partial y}(12\mu U_*), \\ X_5^{(1)} &= \frac{\partial}{\partial y}(c_0\Delta\tilde{U} + c_1\Delta\Delta\tilde{U} + c_2\Delta\Delta\Delta\tilde{U}) - \frac{\partial}{\partial x}(12\mu U_*), \\ X_6^{(1)} &= d_0\Delta\Delta\tilde{U} + d_1\Delta\Delta\Delta\tilde{U}. \end{aligned} \tag{50}$$

where

$$\begin{aligned} a_0 &= 2(\lambda+\mu), \qquad a_1 = -\frac{2(2\lambda+3\mu)(1+\sigma)B}{E}, \\ b_0 &= -24\lambda, \qquad b_1 = 2\lambda+3\mu+\frac{48\lambda(1+\sigma)B}{E}, \\ c_0 &= -12\mu, \qquad c_1 = (2\lambda+3\mu)\frac{\sigma}{1-\sigma}, \qquad c_2 = \frac{(2\lambda+3\mu)(1-2\sigma)}{24\sigma}, \\ d_0 &= 2\lambda+3\mu-\frac{2\lambda\sigma}{1-\sigma}, \qquad d_1 = -\frac{\lambda(1-2\sigma)}{12\sigma}. \end{aligned} \tag{51}$$

One can easily find out that a particular solution of nonhomogeneous equations (40) and (41) may be represented by the formulae

$$\begin{aligned} u_1^{(1)'} &= \frac{\partial}{\partial x}(a_0'\tilde{V} + a_1'\Delta\tilde{V}) - \frac{1}{12}\frac{\partial\psi}{\partial y}, \\ u_2^{(1)'} &= \frac{\partial}{\partial y}(a_0'\tilde{V} + a_1'\Delta\tilde{V}) + \frac{1}{12}\frac{\partial\psi}{\partial y}, \\ v^{(1)'} &= \Delta(b_0'\tilde{V} + b_1'\Delta\tilde{V}); \end{aligned} \tag{52}$$

$$
\begin{aligned}
v_1^{(1)'} &= \frac{\partial}{\partial x}(c_0'\tilde{U} + c_1'\Delta\tilde{U} + c_2'\Delta\Delta\tilde{U}) + \frac{\partial}{\partial y}\left(U_* + \frac{1}{12}\Delta U_*\right), \\
v_2^{(1)'} &= \frac{\partial}{\partial y}(c_0'\tilde{U} + c_1'\Delta\tilde{U} + c_2'\Delta\Delta\tilde{U}) + \frac{\partial}{\partial x}\left(U_* + \frac{1}{12}\Delta U_*\right), \\
u^{(0)'} &= d_0'\tilde{U} + d_1'\Delta\tilde{U} + d_2'\Delta\Delta\tilde{U},
\end{aligned}
\tag{53}
$$

where

$$
\begin{aligned}
a_0' &= -(1+\sigma); \qquad a_1' = \frac{6-13\sigma+5\sigma^2+4\sigma^4}{24(1-2\sigma)^2}, \\
b_0' &= -\sigma, \qquad b_1' = \frac{3-7\sigma+10\sigma^2-8\sigma^3}{24(1-2\sigma)^2}, \\
c_0' &= -d_0' = -\frac{2(1-2\sigma)}{(1-\sigma)^2}, \qquad c_1' = \frac{2-9\sigma+6\sigma^2}{2(1-\sigma)^2}, \\
c_2' &= \frac{(3-2\sigma)(1-2\sigma)}{24\sigma}, \\
d_1' &= -\frac{8-29\sigma+30\sigma^2+8\sigma^3}{2(1-\sigma)^2(1-2\sigma)}, \\
d_2' &= -\frac{3-12\sigma+4\sigma^2}{24\sigma}.
\end{aligned}
\tag{54}
$$

If we now add to the particular solutions (52) and (53) of equations (40) and (41) the general solutions of the corresponding homogeneous equations, we obtain the formulae

$$
\begin{aligned}
u_1^{(1)} &= \frac{\partial W}{\partial x} - \frac{\partial W_*}{\partial y} + u_1^{(1)'}, \\
u_2^{(1)} &= \frac{\partial W}{\partial y} + \frac{\partial W_*}{\partial x} + u_2^{(1)'}, \\
v^{(1)} &= -\frac{\sigma}{1-\sigma}\Delta W - \frac{1-2\sigma}{24\sigma}\Delta\Delta W + v^{(0)'},
\end{aligned}
\tag{55}
$$

$$
\begin{aligned}
v_1^{(1)} &= \frac{\partial Z}{\partial x} - \frac{\partial \omega}{\partial y} + v_1^{(1)'}, \\
v_2^{(1)} &= \frac{\partial Z}{\partial y} + \frac{\partial \omega}{\partial x} + v_2^{(1)'}, \\
u^{(1)} &= -Z + \frac{2(1+\sigma)B}{E}\Delta Z + u^{(1)'},
\end{aligned}
\tag{56}
$$

where

$$W = -\frac{\sigma}{24}\varphi + \frac{1-\sigma}{2(1+\sigma)}[z\bar{\phi} + \bar{z}\phi - \tfrac{1}{2}(\phi_0 + \bar{\phi}_0)],$$

$$W_* = \frac{i}{1+\sigma}(z\bar{\phi} - \bar{z}\phi), \qquad (57)$$

$$Z = z\bar{\Psi} + \bar{z}\Psi + \Psi_0 + \bar{\Psi}_0.$$

Here ϕ_0, ϕ, Ψ_0 and Ψ are the arbitrary analytic functions of z, φ and ω are the arbitrary real solutions of equations

$$\Delta\varphi - \frac{24}{1-\sigma}\varphi = 0, \qquad \Delta\omega - 12\omega = 0.$$

We are to mention that functions $u_1^{(1)'}, u_2^{(1)'}, u^{(1)'}, v_1^{(1)'}, v_2^{(1)'}, v^{(1)'}$ are known since they are expressed by the solution of the boundary value problem A_0 (we have supposed that the solution of this problem is known). Therefore each of formulae (55) and (56) depend on the three arbitrary analytic functions.

It is obvious that the homogeneous boundary conditions for functions $u_1^{(1)}, u_2^{(1)}, u^{(1)}, v_1^{(1)}, v_2^{(1)}, v^{(1)}$ will be reduced to the definite non-homogeneous boundary conditions for analytic functions contained in formulae (55) and (56).

There exists a wide class of boundary value problems for which the corresponding boundary conditions are separated into two groups in the following manner. One group of boundary conditions contains three functions u_1, u_2, v and the other one contains v_1, v_2, u. The boundary value problems of this kind, if we are to find the approximate solutions of the form (39), are split into two independent boundary value problems. Each of them is reduced to the boundary value problem for three analytic functions. This fact, obviously, facilitates the necessary mathematical calculations.

If we use the formulae giving representations of solutions of the initial equations of the spherical shell, we obtain the boundary conditions which contain all six analytic functions. These boundary value problems do not split, in general, into independent ones with less than six sought analytic functions.

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Абстракт—Новая теория упругих оболочек приводит к системе эллиптических уравнений двенадцатого порядка /см. И. Н. Векуа "Теория тонких пологих оболочек переменной толщины". Тр. Тбилиск. мат. ин-та АН Груз ССР, 1965/. Для случая сферической оболочки, решения этих систем уравнений можно точно представить в виде шести функций, удовлетворяющих уравнениям типа $\nabla^2 w + k_i^2 w = 0$/ $i = 1, 2, 3, 4, 5$/, где соответственно $k_1^2, k_2^2, k_4^2, k_5^2$—действительные постоянные, k_3^2—комплексная постоянная, w_1, w_2, w_4, w_5—действительные функции, w_3—комплексная функция. Зти функции можно выразить шестью произвольными аналитическими функциями одной комплексной перемонной. Для пологих сферических оболочек полученные формулы значительно упрощаются. Тотже меуод можно, также, использовать для решения упрощенных уравнений пологих сферических оболочек.